PMM U.S.S.R., Vol.53, No.2, pp.266-268, 1989
Printed in Great Britain
0021-8928/89 $10.00+0.00

PERIODIC SOLUTIONS OF THE ELLIPTICAL THREE-BODY PROBLEM NEAR A TRIANGULAR LIBRATION POINT*

V.P. EVTEYEV and E.M. MUKHAMADIYEV

Two classes of periodic orbits with the period of the plane elliptical three-body problem are investigated. A form of the method of small parameter is employed, and the eccentricity of the problem is used as the small parameter.

Consider the plane elliptical three-body problem /1/. If we assume that the gravitational constant, the major semi-axis and one of the attracting masses are all equal to unity and use the eccentric anomaly E as the independent variable, then the equation of motion of the problem will be written, in complex form, as follows /2/:

$$n^2\{\rho z'' + [e\sin E + 2i(1-e^2)^{1/2}]z' + z\} + 1 = 2\partial_{\bar z} W \tag{1}$$
$$\rho = 1 - e\cos E,\quad n^2 = 1+\mu,\quad z = x + iy$$
$$W = 1/|z-1| + \mu/|z|$$

Here the prime denotes differentiation with respect to E, μ is the second attracting mass $(0<\mu<1)$, and e is the eccentricity which we shall regard as a small parameter.

We shall construct the solution of Eq.(1) in the form

$$z = \sum_{i=0}^{\infty} z_i e^i,\quad \mu = \sum_{i=0}^{\infty} \mu_i e^i \tag{2}$$

where we have taken the libration point $L_4: z_0 = 1/2 + i\sqrt{3}/2$ as z_0.

Substituting series (2) into (1), we obtain the relation

$$\Big(1 + \sum_{k=0}^{\infty}\mu_k e^k\Big)\Big\{(1-e\cos E)\sum_{k=1}^{\infty} z_k'' e^k + [e\sin E + 2i(1-e^2)^{1/2}]\sum_{k=1}^{\infty} z_k' e^k - \sum_{k=1}^{\infty} z_k e^k\Big\} = \tag{3}$$
$$\sum_{1\le m=i+j} b_{ij}\Big(\sum_{k_1+\ldots+k_i+p_1+\ldots+p_j=m} z_{k_1}z_{k_2}\cdots z_{k_i}\bar z_{p_1}\bar z_{p_2}\cdots\bar z_{p_j}\Big)e^m$$
$$b_{ij} = \frac{2}{i!\,j!}\frac{\partial^{i+(j+1)}W(z_0)}{\partial z_0^{\,i}\,\partial\bar z_0^{\,j+1}}$$

Equating the coefficients of like powers in Eq.(3), we construct the corresponding approximate equations.

The first approximation. We obtain the following first-approximation equation for (3):

$$L_0(z_1) = (1+\mu_0)(z_1'' + 2iz_1' - z_1) - b_{10}z_1 + b_{01}\bar z_1 = 0 \tag{4}$$
$$b_{01} = {}^1/_2(1+\mu_0),\quad b_{01} = -{}^3/_4[1 - i\sqrt{3} + (1+i\sqrt{3})\mu_0]$$

where L_0 is a selfconjugate operator.

We find that under these conditions Eq.(4) will have periodic solutions

$$z_1 = c_{11}e^{-i\lambda E} + c_{12}e^{i\lambda E} \tag{5}$$

Substituting (5) into (4), we arrive at the characteristic equation

$$\lambda^4 - \lambda^2 + \frac{27\mu_0}{4(1+\mu_0)^2} = 0 \tag{6}$$

Hence we have reduced the problem of the existence of periodic solutions of the form (5) of Eqs.(4), to the problem of the existence of real roots of the characteristic Eq.(6).

Next we shall investigate the $T = 2\pi/\lambda$ -periodic solutions of Eq.(1), i.e. we shall put $\lambda = 1/2$. Using this assumption we find from (6), that $\mu_0 \approx 0{,}029437$.

**Prikl. Matem. Mekhan.*, 53, 2, 339-341, 1989

Theorem 1. Let μ_0 satisfy relation (6). Eq.(4) will have a periodic solution $z_1 = \alpha\bar{c}_{12}e^{-iE/2} + c_{12}e^{iE/2}$, $\alpha = 1 - i1.632993$ where $\bar{c}_{12}$ is a complex conjugate of the integration constant c_{12}.

Second approximation. The second-approximation equation has the form

$$\begin{gathered}
L_0 z_2 = L_1 z_1, \quad L_1 z_1 = a_0 + a_{-1/2}e^{-iE/2} + a_{1/2}e^{iE/2} + \\
a_{-1}e^{-iE} + a_1 e^{iE} + a_{-3/2}e^{-i3E/2} + a_{3/2}e^{i3E/2} \\
a_0 = [\alpha k_1 + {}^{15}/_4\bar{\alpha}(\mu_0 - 1) + 2k_2]\, c_{12}\bar{c}_{12} \\
a_{-1/2} = {}^3/_4[b_{10}c_{12} + (\alpha - 1 - i\sqrt{3})\,\mu_1\bar{c}_{12}] \\
a_{1/2} = {}^3/_4\alpha b_{10}\bar{c}_{12} + {}^1/_4[11 - 3\bar{\alpha}(1 + i\sqrt{3})]\,\mu_1 c_{12} \\
a_1 = [{}^1/_2 k_1 + \bar{\alpha}k_2 - {}^{15}/_8(1 - \mu_0)\,\alpha^2]\, c_{12}^2 \\
a_{-1} = [{}^1/_2\alpha^2 k_1 + k_2\alpha - {}^{15}/_8(1 - \mu_0)]\,\bar{c}_{12}^2 \\
a_{-3/2} = -{}^5/_4 b_{10}\alpha\bar{c}_{12}, \quad a_{3/2} = -{}^5/_4 b_{10}c_{12} \\
k_1 = {}^3/_8[1 - i\sqrt{3} - (1 - i\sqrt{3})\,\mu_0] \\
k_2 = {}^3/_8(1 + i\sqrt{3})(1 + \mu_0)
\end{gathered} \tag{7}$$

Theorem 2. The sufficient and necessary condition for the inhomogeneous operator Eq.(7) to have a solution is, that its right-hand side be orthogonal for all solutions of the conjugate homogeneous equation.

From Theorem 2 we have the following.

Theorem 3. Eq.(7) has a solution if and only if the equation

$$\begin{gathered}
d_1\mu_1^2 - {}^3/_2 b_{10} = 0 \\
d_1 = {}^3/_4|\alpha|^2 + 11 + {}^3/_4[(1 + i\sqrt{3})\,\alpha + (1 - i\sqrt{3})\,\bar{\alpha}]
\end{gathered} \tag{8}$$

has non-zero real roots.

Eq.(8) has two real roots

$$\mu_1 = \pm 3\,(1 + \mu_0)^2/(10 + 46\mu_0)$$

In what follows, we shall assume that $\mu_1 > 0$. In this case we obtain the relation $\bar{c}_{12} = -c_{12}$, and the solution of the first-approximation equation will take the form

$$z_1 = c_{12}(-\alpha e^{-iE/2} + e^{iE/2})$$

Third approximation. We have

$$\begin{gathered}
L_0 z_3 = L_1(z_1, z_2), \quad L_1(z_1, z_2) = b_0 + b_{-1/2}e^{-iE/2} + b_{1/2}e^{iE/2} + \\
b_{-1}e^{-iE} + b_1 e^{iE} + b_{-3/2}e^{-i3E/2} + b_{3/2}e^{i3E/2} + \\
b_{-2}e^{-i2E} + b_2 e^{i2E} + b_{-5/2}e^{-i5E/2} + b_{5/2}e^{i5E/2} \\
b_{-1/2} = c_{12}\{{}^3/_4\mu_1 A_{-1/2} + {}^3/_4 b_{10}A_{-3/2} + [-{}^1/_2(\alpha A_0 + A_{-1})\,k_1 + \\
{}^4/_3(\alpha\bar{A}_0 - \bar{A}_1 + A_0 + \bar{\alpha}A_{-1})\,k_3 + 20(\bar{A}_0 + \bar{\alpha}A_1)\,b_{10} + 5(\alpha^2 + 14\bar{\alpha})\,k_4 + \\
{}^3/_2\alpha^2(|\alpha|^2 + 1)\,b_{10} - 15(2|\alpha|^2 + 1)\,b_{01}]\,c_{12}^2 + {}^1/_4(1 + i\sqrt{3} - \alpha)\,\mu_2 - b_{10}\alpha\} \\
b_{1/2} = c_{12}\{{}^3/_4\mu_1 A_{-1/2} - {}^3/_4 b_{10}A_{-3/2} + [(A_0 - \alpha A_1)\,k_1 + \\
(A_0 + \bar{\alpha}A_0 - \alpha\bar{A}_{-1} - A_1)\,k_3 - {}^{15}/_2(1 - \mu_0)(\bar{\alpha}\bar{A}_0 + \bar{A}_{-1}) - \\
{}^9/_8(2|\alpha|^2 + 1)\,b_{10} - {}^5/_4(|\alpha|^2 + 2)\,\bar{\alpha}b_{01} + {}^{15}/_4(\alpha + \bar{\alpha}^2)\,k_4]\,c_{12}^2 + \\
{}^1/_4[11 - 3(1 + i\sqrt{3})\,\bar{\alpha}]\,\mu_2 + {}^5/_8\mu_1\alpha - b_{10}\}
\end{gathered} \tag{9}$$

The coefficients $k_3, k_4, A_i\ (i = 0, \pm 1, \pm 2, \pm{}^3/_2, \pm{}^1/_2)$ are complex numbers of the type k_1 and k_2. The remaining coefficients of Eq.(9) do not affect the conditions for the existence of periodic solutions of the third approximation.

We find these conditions by following the arguments used in the second approximation. Using Theorem 2 and assuming that $\mu_2 = 0$, we arrive at the following result.

Theorem 4. The third-approximation Eq.(9) will have a periodic solution if and only if the equation

$$P_0 c_{12}^2 + P_1 = 0 \tag{10}$$

where P_0 and P_1 are complex numbers and c_{12} is the integration constant from the first approximation, has a solution.

Since in this case P_0 and P_1 are not zero, it follows that Eq.(10) will always have two roots. Substituting the numerical values into (10), we obtain

$$c_{12} = \pm\gamma = \pm(0{,}075421 + i0{,}187614)$$

Conclusion. We shall now assume that in the second series of (2) all $(\mu_i = 0, i = 2, 3, \ldots)$. We find the conditions for the existence of periodic solutions of the further approximations just as in the case of the second and third approximation. Beginning with the fourth approximation, these conditions will be reduced to the solvability of the linear equation

$$Pc_{j-2,2}+P_j=0\quad (j=4,5,\ldots)$$

where $C_{j-2,2}$ is the integration constant of the solution z_{j-2} of the $(j-2)$-th approximation respectively. The coefficient $P\neq 0$ will remain general for all j-th approximations.

Thus we have the following theorem.

Theorem 5. Eq.(1) has two families of 4π-periodic solutions which are represented by the first series of (2), and the first approximation has the form

$$z_1=\pm\gamma(-\alpha e^{-iE/2}+e^{iE/2})$$

Similar result can be obtained for the second libration point L_5.

REFERENCES

1. DUBOSHIN G.N., Celestial Mechanics. Basic Problems and Methods. Nauka, Moscow, 1975.
2. EVTEYEV V.P., Periodic orbits of a bounded elliptical three-body problem. Kosmich. Issled. 25, 2, 1987.

Translated by L.K.

PMM U.S.S.R., Vol.53, No.2, pp.268-270, 1989
Printed in Great Britain
0021-8928/89 $10.00+0.00

GROUP-THEORETICAL ANALYSIS OF THE EQUATIONS OF MOTION OF A THIXOTROPIC FLUID*

R.N. BAKHTIZIN

A group-theoretical classification of a system of equations describing a one-dimensional flow of a thixotropic fluid is carried out. Certain invariant solutions are analysed.

1. We shall regard, as a thixotropic fluid, a medium in which an increase in shear stresses leads to a decrease in viscosity due to disruption of internal structure of the medium. Such fluids include asphalt - and paraffin-contg. oils, a number of polymer solutions, clayey solutions, et al.

We describe the flow of a thixotropic fluid, of viscosity μ depending on a single structural dimensionless parameter λ, with help of the models /1, 2/ which can be written, in the one-dimensional case, in the form

$$u_t=(\mu(\lambda)u_x)_x,\quad \lambda_t=\Phi(\lambda,u_x)\tag{1.1}$$

Models of this type are also used when describing the filtration of a viscoelastic fluid /3/.

Let us investigate the group properties /4, 5/ of system (1.1).

When the functions $\mu(\lambda)$ and $\Phi(\lambda,u_x)$ are arbitrary, system (1.1) admits of a three-dimensional algebra L_3 of infinitesimal operators with the basis $X_1=\partial/\partial t,\ X_2=\partial/\partial x,\ X_3=\partial/\partial u,$ corresponding to the shears in t, x, u.

Let us determine under what special conditions imposed on μ and Φ this algebra can be extended. The analysis of the system of defining equations /4/ for (1.1) shows that the following assertion holds:

If $\mu'\not\equiv 0,\ \partial\Phi/\partial u_x\not\equiv 0,\ \partial\Phi/\partial\lambda\not\equiv 0,$ then the algebra extends only for one of the following sets of μ and Φ: 1) μ is an arbitrary function, $\Phi=u_x^{\alpha}f(\lambda)$, where f is an arbitrary function, $\alpha\neq 0$; and the positive basis operator has the form

*Prikl. Matem. Mekhan., 53, 2, 341-343, 1989